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Franz Prettenthaler

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JOANNEUM RESEARCH Forschungsgesellschaft mbH – Institute of Technology and Regional Policy (InTeReg)

Graz Office:
Elisabethstraße 20
A-8010 Graz, Austria
Phone: +43-316-876 1488
E-Mail: intereg@joanneum.at

Vienna Office:
Haus der Forschung, Sensengasse 1
A-1090 Vienna, Austria
Phone: +43-1-581 75 20
E-Mail: intereg@joanneum.at

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Franz Prettenthaler

JOANNEUM RESEARCH, Institute of Technology and Regional Policy
Elisabethstraße 20, 8010 Graz, Austria
e-mail: franz.prettenthaler@joanneum.at
Tel: +43-316-876/1455

Abstract:

Financial aid for the worst-off victims of floods and other catastrophes seems to be a morally unquestioned principle for the allocation of public funds. This paper shows however, that this principle is ambiguous if the decision is viewed as a dynamic choice problem where such resources need to be allocated in two periods: before and after the event takes place (before and after uncertainty is resolved). The literature on social choice suggests that utilitarian principles fare better in such situations. This paper provides a uniform formal framework to relate one such result, namely a multi-profile version of Harsanyis 1955 theorem by Mongin (1994) to another one by Myerson (1981), stated in a somewhat unconventional social choice framework. It shows that the linearity condition, that is met only by welfare functions of the utilitarian type, has a natural interpretation in terms of an equivalence of ex-ante and ex-post evaluation, a concept that is related to but not equivalent with dynamic consistency.

Keywords: Dynamic consistency, ex ante, ex post, egalitarianism, catastrophic risk, risk transfer mechanisms

JEL Classification: G22, H42, I3, Q54

CONTENTS

1 Introduction

It is amazing in how many countries institutional reform in the area of transferring the risk from natural catastrophes by means of state run compensation agencies and/or insurance markets is undertaken right now. If we only have a look within Europe, Belgium adopted a new system in 2005, Germany failed to do so after the 2002 floods after long discussions, in Austria and Italy the debates are still in process, France settled it problems of adverse selection, Romania's obligatory flood insurance starts in 2008. It is also amazing how many economists from diverse origins publish in this area, discussing many desirable features, like the ability to cope with adverse selection and moral hazard, accommodate the right incentives to meet the socially efficient level of prevention, struggle with the problem of high cumulative losses, respect individual freedom of choice and so on and o forth. It is deplorable however, that to my knowledge there are virtually no papers that deal with this issue as what it essentially is: a mechanism design problem. The existence of natural catastrophes gives rise to a huge set of possible states of the world, in which the outcomes for the individual agents differ and are influenced by their own choices. The question of designing a national risk transfer mechanism therefore first amounts to the question of which set of desirable properties of the outcome can be possibly implemented given the actions of the individual agents based on some (partly) private information they have (e.g. their preferences). Now it may well be that all the professional mechanism design people are discouraged by the numerous general impossibility results of their discipline to even enter the debate on such an applied task. Cabrales et al. (2003) and Michel-Kerjan et.al. (2006) are laudable (but very different) examples that this general framework of thinking about the problem is appropriate.

If we accept this, then we should focus first on the properties that the outcome should have by discussing social choice rules (or functions, if they are single-valued). The obvious starting point where to look for appealing social choice rules are those that we can derive from rationality or which are morally intuitive for some other reason. *Dynamic consistency* and *compatibility with egalitarian concerns* are two such properties of this different origin and that are appealing to many. As it happens, *Roger Myerson*, one of the Nobel laureates of 2007 was the first to publish on the fundamental relationship of these two properties in 1981 such that it is also to honour his work and the fruitful impact it could have for down to earth policy making on risk transfer issues, if it only were more broadly spread.

When catastrophic events occur, it seems to be a morally unquestioned principle for the allocation of public funds that the worst off victims get compensation for their losses. This principle is ambiguous if the decision to provide relief for the victims is viewed as a dynamic choice problem where egalitarian policies in favour of the worst off need to be carried out in two periods: before and after the event takes place (before and after uncertainty is resolved). A policy that concentrates on preventing harm ex ante for the worst off considering expected losses may no longer be justified ex post, since other victims may turn out to be worst off then.

It has often been stated that utilitarian policies do not have the same problems to be coherently implemented over time as egalitarian ones. One of the best known arguments, that could already be interpreted in this direction is Harsanyi's 1955 aggregation theorem, that basically states the following: Given VNM individual utility functions and a VNM social welfare function and some Pareto condition, utilitarianism is implied. Whilst the formal argument cannot be questioned, the ethical relevance of the result continues to be debated, for a concise discussion see Mongin and d'Aspremont (1999). However, Myerson (1981) explicitly contrasts Utilitarianism and Egalitarianism in a somewhat unconventional social choice framework and shows that the linearity condition, that is met only by welfare functions of the utilitarian type, has a natural interpretation in terms of an equivalence of ex-ante and ex-post evaluation, a concept that is related to but not equivalent with the concept of *dynamic consistency*. Hammond (1981 and 1983) discusses the question in a richer temporal framework and concludes that ex-post welfare optimality, which is compatible with dynamic consistency, should be the ethically relevant criterion.

Besides a numeric example that illustrates the difficulty for egalitarian policies in a dynamic setting, this paper provides a uniform formal framework to state a result that relates a multi-profile version of Harsanyis theorem by Mongin (1994) to the findings of Myerson (1981) and concludes with some lessons for dynamically consistent policies that want to meet egalitarian concerns before and after catastrophic events.

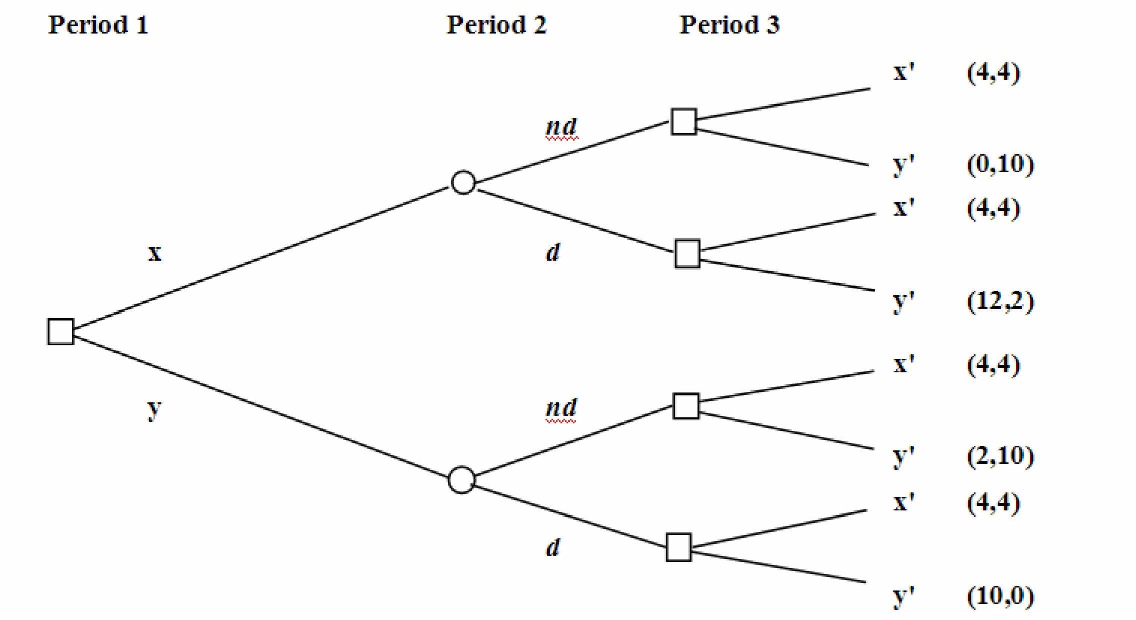
We should justify that we even consider expected utility theory to be informative in a context of catastrophic risk. Not even Oskar Morgenstern (1979) thought that the VNM expected utility theory he initiated with John von Neumann could be useful for low probability events. However, the concern that the multiplicative and additive form of the VNM expected utility functions invite to disregard very low probability high consequence events is often neglected. I believe that this is unfortunate and that for such situations the search for a viable normative theory should continue. But still, dynamic consistency should figure among the properties such an alternative theory exhibits. From dynamic decision theory in the individual context we know how intimately linked dynamic consistency and the independence axiom of expected utility theory are, even though independence can be weakened while keeping dynamic consistency. See e.g. the careful exposition of this question by Cubitt (1996). This is why we think, that the study of dynamic consistency, the independence axiom and its consequences for egalitarian risk policies, i.e. policies that favour the worst off victims is also fruitful in a collective choice context, where dynamic consistency as we will see is per se an ambiguous concept.

2 A numeric example

Let us consider the following numeric example taken from Fleurbaey (1996). Suppose a small society with only two individuals, named A and B, and let us have three periods. The society faces a significant risk of some natural or man made disaster in period 2, where the probability of the disaster is denoted by p_d, whilst p_{nd} denotes the probability for the absence of the event, and $p_d = 1 - p_{nd}$. You can find the associated decision tree in Figure 1, where the respective outcomes after period 3 are given as vectors of utilities $\left(U_A, U_B\right) \in \mathbb{R}$ where U_A denotes the outcome for individual A and U_B denotes the outcome for individual B. In general, the two individuals will have their subjective estimates about how probable the event is, but for the time being, we will take the social decision maker's probability evaluation as given and for convenience we will assume that $d = 0{,}5$. Whilst in period 2 nature has its move (represented in the decision tree in figure 1 by circle nodes) decisions of the society need to be taken in period 1 and 3, represented by squares in the diagram. The decision to be taken in period one concerns e.g. two different building regulations x and y that both affect the individuals in a different way and that have an impact on how well off the individuals are in case of the disaster, say an earthquake or a terrorist attack.

However, in period 3 the society needs to make a decision whether to adopt a policy x' that involves income equalising transfers . In case of the disaster these transfers can be interpreted as a relief-programme for that individual that is worse off. If the disaster did not take place, the transfers can be programme for that individual that is worse off. If the disaster did not take place, the transfers can be thought of as related to help finance some preventive measures. The alternative y' involves no change in the individual's utilities.

Figure 1: The 3-period decision tree associated to the decision problem.



Let us now denote a policy plan by a triple $(a;b,c)$, where $a \in \{x,y\}$ and $b,c \in \{x',y'\}$, where a stands for the policy adopted in period 1, b for the policy that is adopted in period 3 if *nd* prevails and c stands for the policy adopted in period 3 if the disaster d takes place. Note that these policy plans are contingent plans that prescribe a particular action for any state of the world. This makes it easy to calculate the expected utilities of the two individuals for all the respective policy options, as shown in Table 1.

Let us now consider two simple social welfare functions that are of interest for us, the utilitarian one $W_u : \mathbb{R}^2 \to \mathbb{R}$ defined as: $W_u(U_A,U_B) = U_A+U_B$ and the egalitarian one $W_e : \mathbb{R}^2 \to \mathbb{R}$ defined as: $W_e(U_A,U_B) = \min\{U_A,U_B\}$.

If we now apply these two functions to the expected utilities given in column 2 of Table 1, that is to say if we calculate $W_u(EU_A,EU_B)$ and $W_e(EU_A,EU_B)$ respectively, both take their maximal value with the plan (x;y',y'), which will thus be the chosen plan. But suppose now that we embark on this plan and choose indeed option x in period 1 and reconsider the plan in period 3, after the uncertainty has been resolved. This time we apply the welfare functions to the final utilities. As can be verified in Table 2, no matter whether the disaster did or did not occur, whilst W_u chooses y' in both states of the nature, W_e selects x' in both cases. This is in contradiction to our initial plan (x;y',y').

Table 1: Expected Utilities and values of the utilitarian and egalitarian welfare function (ex ante evaluation) associated to every policy plan.

Policy Plan	**(EU_A,EU_B)**	**$W_u(EU_A,EU_B)$**	**$W_e(EU_A,EU_B)$**
(x;x',x')	(4,4)	8	4
(x;x',y')	(8,3)	11	3
(x;y',x')	(2,7)	9	2
(x;y',y')	(6,6)	12	6
(y;x','x')	(4,4)	8	4
(y;x',y')	(7,2)	9	2
(y;y',x')	(3,7)	10	3
(y;y',y')	(6,5)	11	5

Table 2: Histories, policies and respective outcomes with their evaluation by the utilitarian and egalitarian welfare function in Period 3.

Period 1	**Period 2**	**Policy at Period 3**	**Outcomes (U_A,U_B)**	**$W_u(U_A,U_B)$**	**$W_e(U_A,U_B)$**
(x)	(*nd*)	(x’)	(4,4)	8	4
(x)	(*nd*)	(y’)	(0,10)	10	0
(x)	(*d*)	(x’)	(4,4)	8	4
(x)	(*d*)	(y’)	(12,2)	14	2
(y)	(*nd*)	(x’)	(4,4)	8	4
(y)	(*nd*)	(y’)	(2,10)	12	2
(y)	(*d*)	(x’)	(4,4)	8	4
(y)	(*d*)	(y’)	(10,0)	10	0

It seems thus that the utilitarian welfare function is somehow better adapted to dynamic situations than the egalitarian criterion. One might conclude that dynamic consistency and egalitarianism are incompatible. This judgement would be premature though. In fact, there are two questions involved here: The first is whether the two choices before and after uncertainty is resolved are dynamically consistent. We say that an agent chooses in a dynamically consistent way, if she chooses at some time t_n exactly the same option (if still available) as she chose at some time t_{n-x} where the only difference in the set of available options between the two times is that some options available at t_{n-x} are no longer available at t_n. It therefore seems that an agent's decision based on the egalitarian welfare function lacks the property of dynamic consistency.

But we also need to consider a second question, namely whether the welfare evaluation that is employed at the two times is of the type *ex ante* or *ex post*. The ex ante approach is what we applied so far, that is to say, pick a decision node and apply the welfare function to the individual expected utilities at this node. The ex post approach on the other hand consists in first calculating the value of the welfare functions W_u and W_e respectively on basis of the actual utilities in every state, and then calculating the expected value of the welfare functions for each plan. If we apply this method in period 1 to the given example we obtain the expected values for the two welfare function given in Table 3.

Table 3: Expected values of the utilitarian and egalitarian welfare function associated to every policy plan (ex post evaluation).

Policy Plans	**$EW_u(U_A,U_B)$**	**$EW_e(U_A,U_B)$**
(x;x',x')	8	4
(x;x',y')	11	3
(x;y',x')	9	2
(x;y',y')	12	2
(y;x','x')	8	4
(y;x',y')	9	2
(y;y',x')	10	3
(y;y',y')	11	1

Following the utilitarian criterion, we will still choose the plan (x;y',y') and we know that this plan is dynamically consistent. The egalitarian principle on the other hand will choose (x;x',x') or (y;x',x'), and it is easily checked that W_e would choose x' in period 3 in any case and any state of nature. So the choice in period 3 will be the same as planned in period 1. Applying the ex post approach thus also allows choices based on an egalitarian welfare function to be dynamically consistent. We should call the property that choices based on an utilitarian welfare function possess in addition to dynamic consistency then rather "equivalence of ex ante and ex post evaluation". In the following section in which we turn to a more formal study of these question we will introduce this property as a linearity condition on social choice functions.

3 Some Formal Results on Linearity and Independence

We now need to introduce some notation in order to both, restate a result of Myerson (1981) (in a slightly modified version) and of Mongin (1994) and to state our own corollary result based on the two. It will basically relate linearity of Myerson choice functions (MCF) to the independence axiom that is well known from von Neumann-Morgenstern (VNM) expected utility theory, but applied here to a multi profile social choice framework. This is why we need to introduce individual VNM functions, then the rather special kind of MCF's, the formalism of social welfare functionals (SWFL) that aggregate individual VNM utility functions to a social preference relation and we will also make use of an "ordinary" choice function that is based on a binary relation that is an ordering.

3.1 NOTATION

Let $N = \{1,2,...,n\}$ be a finite set of at least 3 integers and $\Xi = \{x_1, x_2, ...x_m\}$ a finite set with at least 2 elements. $p = \{p_1, p_2, ...p_m\} \in \mathbb{R}^m_+, \sum_{j=1}^{m} p_j = 1,$ is called a lottery and $\mathfrak{L}$ is the set of all lotteries. $v_i : \mathfrak{L} \to \mathbb{R}$ is the VNM-utility function of $i \in N$ [1], $\mathfrak{V}$ is the set of all VNM-utility functions and we write $\mathfrak{v}(p) = (v_1(p), v_2(p)...v_n(p))$ for a vector of VNM utilities. Let further be $\succeq \subseteq \mathfrak{L} \times \mathfrak{L}$ a binary relation on $\mathfrak{L}$ and $\mathfrak{B}$ the set of all binary relations on $\mathfrak{L}$. A function $F : \mathfrak{V}^n \to \mathfrak{B}$ is called a Social Welfare Functional (SWFL) and $CP \subseteq \mathbb{R}^n$ is called a choice problem if and only if it is non-empty, closed, convex and comprehensive. $\mathrm{X}\Pi$ is the set of all choice problems, on which we define the so called Myerson choice function (MCF) $f : \mathrm{X}\Pi \to \mathbb{R}^n$. $R \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is an ordering if it is complete, reflexive and transitive. We call $C : \mathrm{X}\Pi \to \mathbb{R}^n$ the choice function on $\mathrm{X}\Pi$ associated to R and defined as $C(S) \subseteq S$, with $C(S) = \{b \in S \mid \forall d \in S : bRd\}, \forall S \subset \mathrm{X}\Pi$.

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[1] A binary relation on the loteries $p,q,r \in \mathfrak{L}$ that satisfies the axioms of ordering, continuity and independence (for definitions see the Appendix) can be represented by such a function $v_i : \mathfrak{L} \to \mathbb{R}$ that takes the so called expected utility form: For $i \in N,\ x_j \in \Xi,\ p_j \in [0,1]$:

$$v_i(x_1, p_1;...;x_j, p_j) \equiv \sum_{j=1}^{m} U(x_j) p_j$$

3.2 THE RESULTS

3.2.1 Theorem 1 (Myerson 1981)

Assume that $X\Pi$ is a convex and compact set of choice problems *CP* and assume that the MCF $f : X\Pi \to \mathbb{R}^n$ is linear and satisfies Strong Pareto, then *f* is utilitarian.

As can easily be verified from the definition of Linearity in Appendix 1, Linearity in this formal framework exactly captures the two notions of ex ante and ex post evaluation employed in our numeric example, where the left hand side of the condition refers to the ex ante evaluation and the right hand side represents the ex post approach. Thus, Linearity does nothing else but demanding that the ex ante approach and the ex post approach yield the same results. This is a more demanding property than mere dynamic consistency as our example showed. The result proofs formally what in our example also could have been a result of well chosen numbers: Ex ante – ex post equivalence together with some pareto principle implies utilitarianism. In other words, if we want that the worst off victims get priority in any social risk policy, ex ante preferences of individuals cannot fully be satisfied, since we need to adhere to ex post pareto optimality if we want to keep dynamic consistency. We will discuss this in the conclusions.

3.2.2 Theorem 2 (Mongin 1994)

If a SWFL $F : \mathfrak{V}^n \to \mathfrak{B}$ satisfies Continuity, Independence, IIA and Strong Pareto, then there exists a vector $\mathfrak{a} = (a_1,...a_n) \in \mathbb{R}^n_+$, such that: $\forall \mathfrak{v} \in \mathfrak{V}^n, \forall p,q \in \mathfrak{L}$: $pF(v)q \Leftrightarrow \sum_{i=1}^{n} a_i v_i(p) \geq \sum_{i=1}^{n} a_i v_i(q)$.

This result is tantamount to saying that in a more sophisticated social choice framework, where individuals have VNM preferences and share their probability evaluations, accepting the independence axiom also for the collective choice under risk implies utilitarianism as well as Linearity did in the different framework of Myerson.

3.2.3 Corollary:

Let $f : X\Pi \to \mathbb{R}^n$ be a MCF; if it can be written as a choice function $C :\ X\Pi \to \mathbb{R}^n$ that is induced by the relation *R*, which is derived from a SWFL $F : \mathfrak{V}^n \to \mathfrak{B}$ of Mongin, then *f* is linear.

Proof see Appendix 2

Since the two results are stated in such different formal frameworks we need this corollary result to be able to state that the Linearity condition and the Independence axiom in the two respective frameworks play the same role in guaranteeing a utilitarian social welfare evaluation.

4 Conclusions

What can we conclude from these results for the question of catastrophic risk mitigation programmes that want to meet the moral intuition that the worst off victims should have priority in getting compensation? The first conclusion is, that a society with such concerns cannot adhere to the independence axiom. We did not fully examine all possible ways how such a society can still adhere to dynamic consistency. But one such way became obvious and that is the second conclusion: A society that gives priority to worst off victims cannot respect ex ante preferences of individuals with regards to risk. Therefore, voluntary individual insurance for losses after catastrophic events will not lead to an ex post outcome that provides enough care for the worst off victims. But such a society also cannot provide the right to individuals to take whatever risk he or she prefers. Concern for the worst off victims after a catastrophic event lead to limitations for ex ante risk preferences. A society that is known to provide care for its members in case of catastrophic events usually has problems in motivating its individuals to take voluntary insurance for catastrophic events. But even if it could motivate its members to do so, it will still have to allocate some extra funds for the worst off victims ex post.

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6 APPENDIX 1

6.1 AXIOMS ON THE MCF f:

Pareto Indifference: A multi-valued function $f: X\Pi \to \mathbb{R}^n$ satisfies Pareto

Indifference $\Leftrightarrow$ $\begin{array}{l} \text{(i) } f(S) \subseteq S, \text{ und} \\ \text{(ii) } \mathfrak{v} = \mathfrak{w} \text{ und } \mathfrak{v} \in f(S) \Rightarrow \mathfrak{w} \in f(S),\ \forall S \in X\Pi,\ \forall \mathfrak{v},\mathfrak{w} \in \mathbb{R}^n \end{array}$

Strict Pareto: A function $f: X\Pi \to \mathbb{R}^n$ satisfies Strong Pareto

$\Leftrightarrow$ $\begin{array}{l} \text{(i) } f(S) \in S, \text{ und} \\ \text{(ii) } v_i \geq f_i(S) \forall i \text{ und } \exists i: v_i > f_i(S) \Rightarrow \mathfrak{v} \notin S,\ \forall S \in X\Pi,\ \forall \mathfrak{v} \in \mathbb{R}^n \end{array}$

Strong Pareto: Pareto Indifference & Strict Pareto

Linearity: A function $f: X\Pi \to \mathbb{R}^n$ is linear $\Leftrightarrow$

$f\left(\lambda S + (1-\lambda)T\right) = \lambda f\left(S\right) + (1-\lambda) f\left(T\right),$

$\forall S, T \in X\Pi,\ \forall \lambda \in [0,1]$, and $\lambda S + (1-\lambda)T \in X\Pi$

Where $\lambda S + (1-\lambda)T$ is defined to be the set

$$\lambda S + (1-\lambda)T = \left\{\lambda c + (1-\lambda)d \,\middle|\, c \in S \text{ and } t \in T\right\}, \forall S, T \subseteq \mathbb{R}^n$$

Utilitarianism: A function $f: X\Pi \to \mathbb{R}^n$ is utilitarian $\Leftrightarrow$ $\exists$ a vector

$\mathfrak{a} = (a_1, \ldots a_n) \in \mathbb{R}^n$ such that: $\begin{array}{l} \text{(i) } \sum_{i=1}^{n} a_i = 1 \text{ und } \forall a_i > 0, \\ \text{(ii) } \mathfrak{a} f\left(S\right) = \max_{\mathfrak{v} \in S} \mathfrak{a}\mathfrak{v},\ \forall S \in X\Pi \end{array}$

6.2 AXIOMS ON THE SWFL f:

Continuity:

$\forall \mathfrak{v} \in \mathfrak{V}^n, F(\mathfrak{v})$ is continuous $\Leftrightarrow$

$\forall p, q, r \in \mathfrak{L},\ \left\{\lambda \in [0,1]: pF(\mathfrak{v})\left[\lambda p + (1-\lambda)q\right]\right\}$ and $\left\{\lambda \in [0,1]: \left[\lambda p + (1-\lambda)q\right] F(\mathfrak{v}) r\right\}$

are closed subsets of [0,1].

Independence:

$\forall \mathfrak{v} \in \mathfrak{V}^n, F(\mathfrak{v})$ satisfies independence that is to say:

$\forall p, q, r \in \mathfrak{L}$, $\forall \lambda \in]0,1]$, $pF(\mathfrak{v})q \Leftrightarrow \left[\lambda p + (1-\lambda)r\right] F(\mathfrak{v}) \left[\lambda q + (1-\lambda)r\right]$.

Pareto-Indifference: $\forall \mathfrak{v} \in \mathfrak{V}^n, \forall p, q \in \mathfrak{L}$, $\mathfrak{v}(p) = \mathfrak{v}(q) \Rightarrow p\ I(\mathfrak{v})q$, where I($v$) is the symmetric part of the relation F(v)

Strict Pareto:

$\forall \mathfrak{v} = (v_1, \ldots, v_i, \ldots v_n) \in \mathfrak{V}^n, \forall p,q \in \mathfrak{L}$,

$v_i(p) \geq v_i(q), i = 1, \ldots, n \ \& \ \exists\ j : v_j(p) > v_j(q) \Rightarrow p\ P(\mathfrak{v})q$

where P(v) is the asymetric part of the relation F(v)

Strong Pareto: Pareto- Indifference & Strict Pareto

Independence of irrelevant alternatives (IIA):

$\forall \mathfrak{v}, \mathfrak{v}^* \in \mathfrak{V}^n, \forall p, q \in \mathfrak{L}$, $\mathfrak{v}(p) = \mathfrak{v}^*(p)\ \& \mathfrak{v}(q) = \mathfrak{v}^*(q) \Rightarrow$

$p\ F(\mathfrak{v})q$ if and only if $pF(\mathfrak{v}^*)q$.

7 APPENDIX 2

Lemma 1: There exists a relation $R \subseteq \mathbb{R}^n \times \mathbb{R}^n$ defined by:

$\forall (b,d) \in \mathbb{R}^n \times \mathbb{R}^n,\ \ \mathrm{b}Rd \ \Leftrightarrow\ \exists \mathfrak{v} \in \mathfrak{V}^n,\ p,q \in \mathfrak{L}:\ \mathfrak{v}(p) = b,\ \mathfrak{v}(q) = d,\ xF(\mathfrak{v})y$ that is an ordering and satisfies the axioms of Continuity°, Independence°, and Strong Pareto°.

Proof of Lemma 1: see Mongin(1994, Lemma 2, p.339)

Lemma 2: Let $R \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be an ordering and let $C:\ \mathrm{XP} \to \mathbb{R}^n,\ C(B) \subseteq B,$ with $C(B) = \{b \in B \mid \forall d \in B : bRd\}, \forall B \in \mathrm{XP}$ be its associated choice function. If R satisfies independence, then C is linear in a generalised meaning.

Proof of lemma 2 : We need to show that $C(\lambda B + (1-\lambda)D) = \lambda C(B) + (1-\lambda)C(D)$.

1) $C(\lambda B + (1-\lambda)D) \subseteq \lambda C(B) + (1-\lambda)C(D)$:

$C(\lambda B + (1-\lambda)D) = \{e \in \lambda B + (1-\lambda)D \mid \forall e' \in \lambda B + (1-\lambda)D,\ eR\,e'\}$

$C(\lambda B + (1-\lambda)D) =$

$= \{e$ of the form $\lambda b + (1-\lambda)d$ for $b \in B, d \in D \mid \forall e'$ of the same form, $eR\,e'\}$

By independence we have:

$\lambda b + (1-\lambda)d\ R\ \lambda b' + (1-\lambda)d$ for $b, b' \in B, d \in D \Leftrightarrow bR\,b'\ \forall b' \in B \Rightarrow b \in C(B)$ and:

$\lambda b + (1-\lambda)d\ R\ \lambda b + (1-\lambda)d'$ for $b \in B, d \in D \Leftrightarrow dR\,d'\ \forall d' \in B \Rightarrow d \in C(D)$. Thus, e can be written as an element of $\lambda C(B) + (1-\lambda)C(D)$. q.e.d.

2) $C(\lambda B + (1-\lambda)D) \supseteq \lambda C(B) + (1-\lambda)C(D)$:

Let $g \in \lambda C(B) + (1-\lambda)C(D)$, thus $g = \lambda b + (1-\lambda)d$ for $b \in B, d \in D$,
thus $b \in C(B)$ and $d \in C(D)$ and from this we have $bRb'\ \forall b' \in B$ and $dRd'\ \forall d' \in D$.

From independence we have: $bRb' \Leftrightarrow \lambda b + (1-\lambda)d\ R\ \lambda b' + (1-\lambda)d$

and: $dRd' \Leftrightarrow \lambda b' + (1-\lambda)d\ R\ \lambda b' + (1-\lambda)d'$.

Thanks to Transitivity of R we have: $\lambda b + (1-\lambda)d\ R\ \lambda b' + (1-\lambda)d$ et

$\lambda b' + (1-\lambda)d\ R\ \lambda b' + (1-\lambda)d' \ \Rightarrow\ \lambda b + (1-\lambda)d\ R\ \lambda b' + (1-\lambda)d' \quad \forall b' \in B, \forall d' \in D$.

Thus, $g \in C(\lambda B + (1-\lambda)D)$. q.e.d.

Lemma 2 is jointly established by statements 1) and 2). The reasoning was made for $\lambda \neq 0$, the case $\lambda = 0$ is trivial.

Lemma 3: Let there be given a preference relation $R \subseteq \mathbb{R}^n \times \mathbb{R}^n$ that is an ordering and satisfies the axioms of Continuity°, Independence° and Strong Pareto°. Then the choice function associated to it, $C:$ XP $\rightarrow \mathbb{R}^n$, is utilitarian in a generalised meaning, that is to say, there exists a vector $a = (a_1,...a_n) \in \mathbb{R}^n$ such that :

(i) $\sum_{i=1}^{n} a_i = 1$ and $\forall a_i \geq 0$, et

(ii) $w \in C(B) \Leftrightarrow aw \geq aw'$, for $\forall w' \in B$

Proof of lemma 3: Applying the expected utility theorem (as defined in Fishburn 1982, chap.1) we can conclude that R can be represented by a function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $V = \sum a_i v_i + b$.

That comes down to saying that there exists a vector $\mathfrak{a} = (a_1,...a_n) \in \mathbb{R}^n_+$ such that

$\forall \mathfrak{v}, \mathfrak{v}' \in \mathbb{R}^n$, $\mathfrak{v} R \mathfrak{v}' \Leftrightarrow \sum_{i=1}^{n} a_i v_i \geq \sum_{i=1}^{n} a_i v_i'$. Thus, the choice function induced by R satisfies (ii). q.e.d.

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JOANNEUM RESEARCH Forschungsgesellschaft mbH – Institute of Technology and Regional Policy (InTeReg)

Graz Office:
Elisabethstraße 20
A-8010 Graz, Austria
Phone: +43-316-876 1488
E-Mail: intereg@joanneum.at

Vienna Office:
Haus der Forschung, Sensengasse 1
A-1090 Vienna, Austria
Phone: +43-1-581 75 20
E-Mail: intereg@joanneum.at